How Polynomials Were Solved Throughout History

(Topic: Solving Polynomial Equations)

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Introduction

When we consider the origin of mathematics, it is clear that early developments largely arose from necessity: A farmer wanting to count the amount of sheep on his field, a city wanting to know what area a certain property might have, a government needing a way to collect taxes, ...

Thus we quickly arrive at numbers and simple arithmetic. And from there, polynomials are only a short way away. It might come as a surprise, then, that some of the very basic-seeming questions surrounding polynomials were only answered quiet recently.

For most of us, polynomials have been a constant companion throughout our high school education. Traditionally one starts with solving linear equations, followed by systems of linear equations, later one moves on to quadratics – and then? Then it just stops. But why is that? Why do students have to memorize the solution formula to a quadratic but not to a cubic equation? Is there no such formula? And if so, why not?

All of these questions I want to address in this paper and at the same time relate the relevant discoveries to their historic origins. Personally, I regard this project as a sort of clean-up of the many questions that school left unanswered.

As far as sources go, I mainly relied on three books

- J. Stillwell, *Mathematics and its History*, Springer, 2020
- H. Wußing, *4000 Jahre Algebra*, Springer, 2014
- H. Wußing, *6000 Jahre Mathematik (Band 1)*, Springer, 2013

as well as on the website `https://mathshistory.st-andrews.ac.uk/` and a couple of others. Both figures were produced with tikz.

What is a polynomial?

Before we can talk about solving a polynomial, we first have to establish exactly what we mean by it. In a very simple sense a polynomial is a sum of the form

$$a_n x^n + \ldots + a_1 x + a_0$$

where $a_0, \ldots, a_n$ are the *coefficients* and x is the *variable/unknown*, moreover n is the *degree* of the polynomial (assuming $a_n \neq 0$). All the possible x values which map to zero are called *roots/zeros*. The coefficients are usually known and a polynomial is evaluated by plugging in numbers for x. The choice of coefficients is always dependent on a set of numbers. In school, this is most likely the set of rational numbers $\mathbb{Q}$, but in theory the techniques that are learned there could also be applied to the set of real numbers $\mathbb{R}$ which include $\pi, \mathrm{e}, \sqrt{2}, \ldots$

We will see later in this paper that the search for a closed solution formula has historically been one of the main forces behind introducing "higher" sets like the complex numbers $\mathbb{C}$.

Today, we like to think of polynomials as functions from $\mathbb{R}$ to $\mathbb{R}$, so that we can easily graph them (see figure 1). This makes seeing their characteristics much easier. But it is important to realize that these tools were not used in ancient cultures.

In modern mathematics, a more abstract definition can be given using the concept of a polynomial ring. This, however, is beyond the scope of this project which will mainly be dealing with historic approaches.

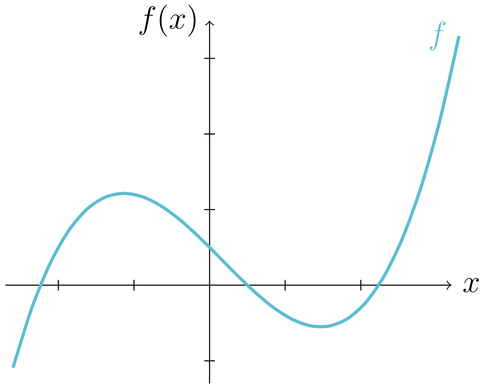

Figure 1: A part of the graph of the polynomial $\frac{1}{5}x^3 - \frac{1}{10}x^2 - x + \frac{1}{2}$

It should be noted, too, that this paper will use modern mathematical notation throughout. This is, of course, a rather recent development, originating in the 16th century.
Many of the cultures that will be referred to here had their own symbols for numbers and arithmetic operations, some did not use base 10 like we do. And almost all of them were using more basic sets like $\mathbb{N}$ or $\mathbb{Q}^+$ to do their calculations.

(Systems of) Linear Equations

When the first societies developed in Mesopotamia (around 1830 BCE), the construction of roads, temples and palaces as well as the division of crop fields or the observation of astronomical phenomena made the calculation of diagonals, circumferences and volumes necessary and thus translated geometrical problems into algebraic ones.

Although the Babylonians did not use symbolic mathematics like we do today and although they had no concept of proofs, it was common to spread knowledge through examples. Since their preferred way of writing was on clay tablets (which are much more durable than papyrus), we know quite a lot about their methods of calculation and their society in general. One such technique that has stood the test of time is the *method of false position*:

Given a linear equation like

$$x + \frac{x}{4} = 15$$

the first step would be to have a guess at a solution. Here $\bar{x} = 4$ is a good choice because its outcome results in an integer $4 + \frac{4}{4} = 5$. Since the result is not the solution of the original equation, but three times smaller, multiplying $4 \cdot 3$ yields the correct result $x = 12$ which can be checked by plugging it back into the left side and calculation the corresponding right side $12 + \frac{12}{4} = 15$. This time the right side equals the original one.

Nowadays we would write the equation differently as $\frac{5}{4}x = 15$ and cross multipy by $\frac{4}{5}$ to get $x = \frac{4}{5} \cdot 15 = 12$, but the method of false position avoids dealing with the inherent fraction.

This method was known throughout many cultures and stayed popular till the 17th century when the advent of algebraic symbols slowly made its usage obsolete.

A closely related problem is the solution of a system of linear equations. Such a system has the form

$$\begin{array}{ccccccc} A_{11}x_1 & +A_{12}x_2 & +\dots & +A_{1n}x_n & = & b_1 \\ A_{21}x_1 & +A_{22}x_2 & +\dots & +A_{2n}x_n & = & b_2 \\ & & & & \vdots & \\ A_{n1}x_1 & +A_{n2}x_2 & +\dots & +A_{nn}x_n & = & b_n \end{array}.$$

Usually given two or three equations and a similar amount of variables, the goal is to find the combination(s) of numbers $x_1, x_2, \ldots$ that solve all equations at once.

The Babylonians mainly used the case of two unknowns and two equations. But a much better system was developed during the Han dynasty (205 BCE – 220 BCE) in China. The Chinese algorithm was essentially what we would today call *Gaussian elimination*. The method consists of subtracting the necessary amount of each equation from the one below to create the triangular form

$$\begin{array}{rrrrcl} A'_{11}x_1 & +A'_{12}x_2 & +\ldots & +A'_{1n}x_n & = & b'_1 \\ & A'_{22}x_2 & +\ldots & +A'_{2n}x_n & = & b'_2 \\ & & & & \vdots & \\ & & & A'_{nn}x_n & = & b'_n \end{array}.$$

After that is done, the solution is found by solving the resulting linear equations from the bottom upwards and substituting in the previous results. In order to keep track of the many coefficients and to perform simple row operations (as we would call them today), a special device called a counting board was utilized.

Quadratic equations

Perhaps the most famous polynomials are the quadratics which were usually encountered in problems related to the calculation of areas. As early as 2000 BCE the Babylonians could solve a pair of equations of the form

$$\begin{array}{rcl} x+y & = & p \\ x \cdot y & = & q \end{array}$$

which is really just the quadratic $x^2 + q = px$ in disguise. The two solutions to this equation were than the values of x and y, although both had to be positive since the Babylonians did not allow negative numbers. Their method of constructing the solution was according to what would today be referred to as Vieta's formula.

At this point it is worth noting that the Babylonians definitely knew the Pythagorean theorem. And for both Pythagoras' formula and the solution to the quadratic above, it is necessary to be able to calculate square roots numerically. Notably they possessed an algorithm to calculate irrational square roots to an arbitrarily precise decimal place – *Heron's method.*

A first general solution formula was given by Brahmagupta (around 650 CE) who solved the quadratic $ax^2 + bx = c$ through

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}.$$

This formula is very similar to the one we use today, but it again omits the negative solution.

While Brahmagupta's and the Babylonian formula are certainly correct, one could speak of a lack of rigor. In both cases there is no discussion of the formula's origin or a precise study of the irrational numbers which appear as results of them. In fact, Brahmagupta himself gives an equivalent formula to the one above only a few lines later in the same text. This raises quite a few doubts about how well he even understood this problem.

Rigorous proofs were introduced into mathematics by the ancient Greeks. The most important book to come out of this time period is Euclid's *Elements*. They contain a collection of the mathematical knowledge at the time, a clear structure and a proof for every theorem. Euclid (around 250 BCE) did not come up with these proofs himself. He rather served as a kind of editor, collecting the works of former generations and bundling it all up in this one book.

Euclid gives a geometric proof (as was typical for the Greeks) of a general quadratic in cases of a positive root. He uses parallelograms to do so. The algebra that is implicitly happening in his proof is far from obvious, however, and it is very likely that Euclid was not aware of it at all, for than he would probably have given a simpler geometrical proof like the one shown below.

A real transition towards a more algebraic understanding of the problem at hand is the solution method given by al-Khwārizmī (around 800 CE) who was still working with a geometric construction, but this time a much more intuitive one: He interpreted the square term x^2 as a literal square with side length x and the other terms like $5x$ as rectangles with side lengths 5 and x. His method of solving the quadratic is than achieved by "completing the square" which is still taught in schools today as a softer introduction to possible solution methods for quadratics.

Before talking about the geometric interpretation, I would first like to follow his reasoning algebraically – as that is the way we are most used to dealing with these kinds of equations nowadays. What al-Khwārizmī realized is that under the right circumstances certain quadratics can easily be solved. In particular, he wanted to take advantage of the "nice" situation that occurs if a quadratic is a perfect square like $x^2+2x+1 = (x+1)^2$ which means that it factors neatly.

His method can now be viewed as a way to brute force this very scenario. We start off with a general quadratic like

$$x^2 + 10x = 39.$$

The expression on the left turns into a perfect square if we add 25 to both sides of the equation. Than we get

$$(x + 5)^2 = x^2 + 10x + 25 = 64.$$

By taking the square root, we only have to solve the linear equation $x + 5 = 8$ to get $x = 3$. It is easy to memorize that the factor that needs to be added on to complete the square on the left side is half of the coefficient of $10x$ squared. But where does that come from? The beauty lies within the geometry:

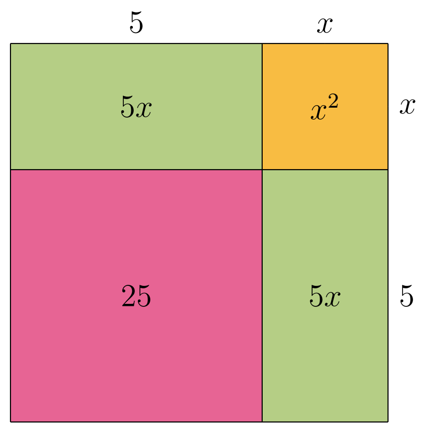

Figure 2: al-Khwārizmī's graphical solution to $x^2 + 10x = 39$

We start off by interpreting x^2 as the area of a square and $10x = 5x + 5x$ as the area of two rectangles of area $5x$. If we join the two rectangles to the square at the side of length x, we get the green and orange reversed L shape in figure 2. To "fill this square up", an additional area of 25 is now apparently needed. So we add 25 to both sides and see the right side of the equation as a square of area $64 = 8 \cdot 8$ and the left side as a square of area $(x + 5) \cdot (x + 5)$. Thus we get $x = 3$ if we solve the underlying linear equation. Geometrically it is clear, why al-Khwārizmī "forgets" about the negative solution $x = -13$: because no negative lengths and areas exist.

If negatives are to be avoided consistently one in general has three types of quadratics to work with

$$x^2 + ax = b, \qquad x^2 = ax + b, \qquad x^2 + b = ax.$$

Each one than has its own solution mechanism.

To end this section, I should quickly talk about how the actual calculation of square roots was dealt with. As I have already mentioned, the cultures that focused very much on the computational aspect of mathematics (like the Babylonians and the Indians) did not give much thought to the more abstract study of the numbers they produce. Although it should be noted that Heron's method to calculate the decimal expansion of any (irrational) square root is quite an extraordinary result in applied mathematics. In essence it is equivalent to Newton's method applied to the special case of finding the positive zero of $f(x) = x^2 - a$.

The Greeks did consider these "new" numbers. Mathematicians like Pythagoras were certainly aware of the proof that Euclid gives that $\sqrt{2}$ is irrational. This shook their perceptions of mathematics quite a bit since it had been assumed that the entire number system was built on integers ("All things are number" was the slogan of the Pythagoreans, numbers to them were fractions of integers). Their disbelief at these new numbers was even captured in their name: irrational numbers. In fact, one legend even goes so far as to suggest that when Hippasus came up with the proof of the irrationality of $\sqrt{2}$ during a boat ride, he was subsequently thrown overboard by his fellow Pythagoreans to drown.

It is worth noting that to most people, the concept of irrational numbers seems far less crazy today than it seemed to the Pythagoreans. That is mainly due to the fact that we are taught that the symbol $\sqrt{2}$ represents the *decimal number* (this is where the dogma comes in) $1.41321456\ldots$ But to first conceive of such an idea and to add these kinds of numbers into your traditional number system must have taken quite a lot of bravery.

Cubic and quartic Equations

We now leave the world of the ancient Greeks and the Babylonians and jump into 16th century Italy. That is indeed the amount of time it took for a complete solution formula for these higher degree polynomials to be found. Although it should be noted that in special cases, solutions were found by, among others, the Persian mathematician Omar Chayyām (around 1070).

The story surrounding the discovery of the cubic formula and its discoverer is quite an interesting one, so I want to describe it here in a few short words. To set the scene, it is important to note that today's philosophy of *publish or perish* in academia was not the norm back then. In fact, it was quite common for mathematicians to keep their discoveries to themselves. This had its good reasons: If you were a math professor in 16th century Italy, a possible opponent could at any time challenge you to a sort of "mathematical duel". The two competing mathematicians exchanged lists of questions and agreed to meet on a given date to present their solutions publicly. If your challenger managed to solve your challenges better than you solved his, he took over your position. In this sense, your theorems were your figurative dueling weapons.

And so it happened that in 1515 the professor of mathematics Scipione del Ferro solved an equation of the type $x^3+ax=b$ algebraically (negative numbers were still avoided which, like with squares, leads to even more different cases). He handed his finding on to his student Maria Fiore who later left for Venice. To make a name for himself there, he challenged the town's arithmetician Niccolò Tartaglia to an open duel. Fiore's 30 questions consisted of finding many different solutions to cubics like "A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is the profit?" Tartaglia actually managed to find and prove the formula that del Ferro had found (according to some legends) on the night before the duel and was able to win because of it.

Word of this spread to the polymath Girolamo Cardano who had long been trying to solve the cubic equation himself, but to no effect. Cardano begged Tartaglia for his formula and likely swore a holy pledged never to tell anyone else. Tartaglia reluctantly agreed and handed over his formula in the form of a poem, but not the proof which he had found. Cardano than set out to find a prove himself and upon learning that del Ferro had proved the same result earlier, he published the formula and a few other cases of the cubic that he had managed to prove as well.

Tartaglia, understandably, was furious upon finding out. The two ended up fighting about this result for the rest of their lives. Cadano's *Ars magna*, which included all of these formulas and the quartic one which a student of Cadano's called Ludovico Ferrari had found, quickly turned into the standard text book on algebra for many years to come. And thus even today, the solution for the cubic and quartic equations are sometimes wrongly accredited to Cadano.

Before I get into Cadano's proof, I should mention that I will not spend much time talking about quartic equations. This is for two reasons: (i) Their nature and derivation is very similar to the cubic one and (ii) the formulas get even messier and would take up more space than they are worth.

I now want to describe the proof that Cadano gave algebraically. It is worth noting, though, that Cadano thought about this in a very geometric sense, similar to the work by al-Khwārizmī above, but I will reason through it algebraically as that is the form we are more accustomed to nowadays.

First we start off with a general cubic of the form

$$aX^3+bX^2+cX+d=0.$$

We will now prove that every general cubic can be rewritten in the form $x^3+px+q=0$. To do this, we divide both sides by a and end up with $X^3+\frac{b}{a}X^2+\frac{c}{a}X+\frac{d}{a}=0$. Now we perform a shift by substituting $X=x-\frac{b}{2a}$. The problem can than be rephrased in the variable x as

$$\begin{aligned} & \left(x-\tfrac{b}{2a}\right)^3 + \tfrac{b}{a}\left(x-\tfrac{b}{2a}\right)^2 + \tfrac{c}{a}\left(x-\tfrac{b}{2a}\right)+\tfrac{d}{a} &= 0\\ \Leftrightarrow\quad & x^3 + \left(\tfrac{c}{a}-\tfrac{b^2}{3a^2}\right)x + \left(\tfrac{2b^3}{27a^3}-\tfrac{bc}{3a^2}+\tfrac{d}{a}\right) &= 0\\ \Leftrightarrow\quad & x^3 + p\,x + q &= 0 \end{aligned}$$

where we introduce $p:=\frac{c}{a}-\frac{b^2}{3a^2}$ and $q:=\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a}$ to clear up the clutter and to emphasize that these are still just coefficients. So now we can just solve the simpler problem $x^3+px+q=0$ and than undo the shift from above through $x=X+\frac{b}{2a}$ and thus arrive at the solutions for the general cubic $aX^3 + bX^2 + cX + d = 0$.

Now we can assume, without loss of generality, that the cubic equation is in this simpler form $x^3 + px + q = 0$.

To find its solution, we choose to rewrite the variable as a sum of two numbers $x = u + v$. This seems like it makes the whole problem even more difficult because we just introduced another two unknowns, but – as we will see in a moment – if we choose u and v correctly, we can achieve a similar kind of brute forcing into a perfect cube like we did with the quadratic and the perfect square. So if we assume just that, we get

$$\begin{array}{rrcl} & (u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 & = & u^3 + 3uv(u+v) + v^3 \\ \Leftrightarrow & (u+v)^3 - 3uv(u+v) - (u^3+v^3) & = & 0 \\ \Leftrightarrow & x^3 - 3uvx - (u^3+v^3) & = & 0 \end{array}$$

If we now compare coefficients between

$$x^3 \; -3uv \; x \; -(u^3+v^3) = 0 \quad \text{and} \quad x^3 + p\,x + q = 0,$$

we get three equations that have to be fulfilled:

$$\begin{array}{ccc} uv & = & -\frac{p}{3} \\ u^3 + v^3 & = & -q \\ x & = & u + v \end{array}.$$

This is again a quadratic in disguise, similar to the case of the perfect square in the previous section. In this instant we get $(v^3)^2 + qv^3 - (\frac{p}{3})^3 = 0$ in the variable v^3, as a little bit of algebra shows. If we apply the quadratic formula, we end up with

$$v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \quad \Leftrightarrow \quad v = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

If we look back at the three equations above, it is obvious that u and v enter symmetrical into every one of them and thus if we had solved for u instead, we would have arrived at the same result.

The solutions are still directed by $x = u + v$ and through the plus/minus sign, we have a total of three options for possible x values. By substituting all three back into the original equation, we find that only the case of different signs (u with $+$ and v with $-$ or the other way around) is an actual solution to the equation. This happens to be the case because some of the steps above are implications and not equivalencies. Thus the answers we end up with are only possible values for a solution, but they don't have to be one.

Thus, finally, we end up with the general solution formula to a cubic as

$$x = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

This might come as a bit of a surprise since, as was well known back than, cubic equations can have up to three roots and the formula now seems to suggest that there is only one. And while this formula is certainly not as easy to remember as the one in the square case, it has two other problems that are unavoidable and quiet displeasing at first sight:

Firstly, neat solutions can appear much more chaotic than they actually are. To illustrate this, we consider $x^3 - 6x - 40 = 0$. If we plot this function, we see that it only has one root and a quick calculation shows that the root is indeed, as the graph would suggest, at $x = 4$. Let us now see, if the cubic formula agrees. Plugging in the necessary numbers yields

$$\sqrt[3]{20 - \sqrt{392}} + \sqrt[3]{20 + \sqrt{392}}.$$

This is in fact equal to four, but seeing that is certainly not an easy task.

But there's an arguably even worse problem to the formula. To show this, we consider $x^3 - 6x - 4 = 0$ with the three real roots $x_1 = -2$, $x_2 = 1 - \sqrt{3}$, $x_3 = 1 + \sqrt{3}$ (as plugging into the equation would show). In this case, the cubic formula would yield

$$\sqrt[3]{2 - \sqrt{-4}} + \sqrt[3]{2 + \sqrt{-4}}.$$

The problem here is obvious: What are we supposed to do with the square roots of negative numbers? Mathematicians like del Ferro were certainly very surprised to see things like this happening. And Cadano was the first to take the necessary steps towards calculations with complex numbers, although he was not aware of their underlying properties but rather choose to calculate with them as a necessity. He did this because he noticed that the cubic formula would sometimes result in an expression like $(5+\sqrt{2}) \cdot (5-\sqrt{2})$. Naively calculating this the way we usually deal with mixed products actually yields a nice result: $(5+\sqrt{2}) \cdot (5-\sqrt{2}) = 25 - \sqrt{-2} \cdot 5 + \sqrt{-2} \cdot 5 + (\sqrt{-2})^2 = 25 - 2 = 23$.

This case shows very nicely how the introduction of complex numbers only seems rational at this point. Just like the solution of the quadratic formula revealed a higher set of numbers (the reals), working with cubic or quartic equations lastly runs into similar problems if we do not switch to a higher set of numbers (the complex numbers). It is probably easier for us to accept this today than it might have been for Cadano or Tartaglia in the 16th century. Back then, a lot of willpower was needed to go against everything that was standard mathematics and define something like $\sqrt{-1}$, just like defining something like $\sqrt{2}$ seemed to be quite a mental challenge for the Greeks. And just like the irrationals before them, the complex numbers were not really named reasonably as a result of this. In fact, nowadays we follow René Descartes' naming convention and call expressions like $a\sqrt{-1}$ *imaginary numbers*. At this point, it might be worth wondering about how different we really are from the Pythagoreans after all and how long we might still have to go until complex numbers are more intuitively accepted by society.

Higher order equations and more

It was thought for a long time that the solution formula to the cubic and quartic equations would be a giant milestone for mathematics. In the end, we saw that that was not quite the case. The formula for cubics is already very messy and the one for quartics even more so. That is the reason why they were never really paied much attention.

Perhaps the most notable consequence of these considerations were not even the formulas themselves, but rather that they opened the door into the world of complex numbers; that they showed without a doubt that their introduction was unavoidable to solve more difficult problems. The problem of the negative square roots in the cubic formula was dubbed the *casus irreducibiles*. Some later work showed that this case could be avoided through the use of trigonometry. But really this detour was only needed by those mathematicians that were not brave enough to give this new concept a chance.

After reading the previous section, it should come as no surprise that the proper introduction of complex numbers followed shortly after del Ferro, Tartaglia and Cardano. The discoverer once more came from Italy (around 1572), his name was Rafael Bombelli, an engineer who knew the works of Cardano, but thought that his explanations were not good enough. He decided to publish his own algebra text book to improve on all the algebra books he knew. Bombelli ended up being the last of the great Italian renaissance algebraists. The focus than shifted towards French mathematicians like René Descartes.

But complex numbers were not properly considered as a serious tool until much later. Euler famously proved the identity

$$\exp(ix) = \cos(x) + i\sin(x)$$

which hints at the great potential that lies within complex numbers to express rotations and periodic movement quite naturally (this is also the reason that trigonometry leads to the answer in the above case of the casus irreduzibiles). Nowadays it is hard to picture modern mathematics without complex numbers. And almost everyone who pursues some sort of higher math education will be introduced to them sooner or later.

Perhaps the most surprising and beautiful result that the complex numbers entail is the *Fundamental theorem of algebra* (its name hints at its importance) which was first proved by Gauss in his doctoral thesis in 1799. It states that every non-constant polynomial of degree n with complex coefficients can be fully factorized into n linear factors. This means that over $\mathbb{C}$ every (non-constant) polynomial has exactly n roots.

In a way, this sort of answers a question which many of the mathematicians that we have talked about likely had: Why do some polynomials have n solutions while others do not? The answer is now simply that if their solutions are not real, they have to be imaginary (quite contrary to what their name would suggest).

Another result that is implied is that we will not ever run into "unknown" terms in any higher degree polynomials that we deal with. That is to say: We will not have to go over to an even higher set to account for some lack of solutions.

All of this sounds very positive and nice. And I can certainly say from my personal experience with both real and complex polynomials that working over $\mathbb{C}$ is almost always a more pleasant experience. So one might be forgiven to think that, equipped with our new number system, we might also be able to tackle the problem of finding higher order solution formulas for quadratics once and for all.

Surprisingly, the definitive answer is that this task is hopeless. In 1799 Paolo Ruffini proved that finding a general solution formula to polynomials of degree higher than four is impossible. His proof had gaps however, gaps that Niels Henrik Abel closed 25 years later by giving a correct proof. This result is today referred to as the *Abel-Ruffini theorem* in honor of their work. While a general formula is impossible, in certain special cases a concrete solution formula may be given.

A rather simple proof of the Abel-Ruffini theorem can now be achieved by viewing the problem through the lens of *Galois theory*. However, the proof that Abel gives does not use Galois theory which was developed only shortly after and is thus much more complex (and too long to include here). But the simplicity of the solution in this case goes to show the strength that these deeply abstract, algebraic considerations can have.

All of this might seem like a somewhat depressing conclusion to the question from the beginning of the paper. But nowadays through the usage of computers, we have found many methods to compute roots to an arbitrary precision; not unlike the Babylonians' Heron method. In this sense, we are only ever as far away from a given solution as *we* choose to be.

To end this paper, I think it is certainly the right place to highlight again how old the study of polynomials is. Essentially, they have been studied for as long as math has been around. And for most of that time, finding a solution was a very hard task to handle. A journey that started over 4000 years ago has sort of found its conclusion only within the last 200 years or so.

Of course, there are still more things to be learned about polynomials, too. And while the motivation behind their usage might now have much more complex origins (eigenvalues, taylor approximation, Weierstrass approximation theorem, ...), the algebra that surrounds their solutions has essentially stayed the same.

Their simplicity on the one hand and the complexity which certain questions about them entail on the other have made them into some of the most rewarding objects of mathematical study throughout history.

Closing remarks

Lastly, I should talk about my personal experience with polynomials a little bit. I think the way that most of us are introduced to them in school makes them seem quiet dull. Solving linear and quadratic equations is practiced time and time again algebraically until it turns into a kind of strange ritual that one knows to perform. I think many former high school students could still be able to recite the quadratic formula today. But as far as understanding goes, I am not so sure.

To me, plugging in values into a formula is not really at the heart of mathematics. The wonderful role of math is finding the formula so that the exact calculations turn into a mere algorithm – that is what we have computers for! I think we tend to forget that the mathematics which we have at our disposal today resulted from revolutionary discoveries. Most of the basic calculations we do, the ones we might call "simple arithmetic", are only so simple to us because we are brought up in this algebraic world where we can translate a real world problem into a few symbols on a sheet of paper, solve the problem there and than go back to the real world.

In researching this paper, I found it really hard to imagine not using negative numbers, for instance. They seem like the obvious conclusion to an equation like $5x + 10 = 5$. But how could I know what it must have been like to be the first person to allow for them? I was introduced to the concept of integers as a fifth grader after all.

I think exploring the way that the sets we use today came about historically gave me a bigger appreciation for them. One has a very clear motivation to define exactly what a real number is if a square root is to be taken of a number like 2. And in a similar sense – and one that I can relate to much more than the example with the negatives –, the introduction of the complex numbers seems like a stretch only as long as you understand that this process of expanding sets to fit the problems of your time is a quite natural one; one that has been practiced for thousands of years.

Lastly I noticed how much I have changed my view on polynomials since I left school. Seeing their connection to all of these ancient cultures and mathematicians is quite humbling in a way. And, to me, that is what studying history does. It shows where we are relative to what has come before and it gives us a glimpse of how much must have happened to get us to where we are today.